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A scaling approach to the existence of long cycles in Casimir boxes

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Abstract

We analyze the concept of generalized Bose–Einstein condensation (g-BEC), known since 1982 for the perfect Bose gas (PBG) in Casimir (or anisotropic) boxes. Our aim is to establish a relation between this phenomenon and two concepts: the concept of long cycles and the off-diagonal-long-range-order (ODLRO), which are usually considered as an adequate way to describe standard BEC on the ground state for cubic boxes. First, we show that these three criteria are equivalent in this latter case. Then, based on a scaling approach, we revise the formulation of these concepts to prove that the classification of g-BEC into three types I, II, III, implies a hierarchy of long cycles (depending on their size scale) as well as a hierarchy of ODLRO which depends on the coherence length of the condensate.

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1. Introduction

1.1. About Bose–Einstein condensation

Bose–Einstein condensation (BEC) predicted in 1925 was discovered first in superfluid ^{4}He in 1975 in deep-inelastic neutron scattering experiments (see [Z-B] for historical remarks). Since 1995 it has been attracting a lot of attention from theoretical and mathematical physicists motivated by experiments with ultra-cold gases in traps [L-S-S-Y].

Recently experimentalists have discovered some new peculiarities of Bose–Einstein condensation in very anisotropic traps [M-H-U-B]. This may imply in certain cases a (what they called) *fragmentation* of the condensate. In fact this phenomenon was predicted a long

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time ago by Casimir [C]. Then it was carefully studied by mathematical physicists and now it is known under the name of *generalized* Bose–Einstein condensation (g-BEC) *à la van den Berg-Lewis-Pulé* [Z-B]. After the first publication by van den Berg and Lewis in 1982 [vdB-L], a set of articles treating different cases of Casimir's anisotropic box, boundary conditions and external potentials [vdB, vdB-L, vdB-L-P, vdB-L-L] has appeared. They classified Bose–Einstein condensation into three types. If a finite/infinite number of one-particle kinetic-energy quantum states are macroscopically occupied, then the Bose gas manifests g-BEC of type I/II. If there are no states macroscopically occupied, although in the thermodynamical limit a macroscopic number of particles is accumulated in the ground (zero-mode) state, then the Bose gas manifests g-BEC of type III.

It is worth noting that not only the box anisotropy or boundary conditions, but also the *interaction* between particles is able to modify the type of the condensate, see [M-V, Br-Z].

In 1953 Feynman [F] introduced a concept of *cycles* by rewriting the partition function of the boson gas using the Bose statistic. In the Feynman–Kac representation, cycles are closed random Wiener trajectories. This was a mathematical framework for the path-integral formulation of the quantum statistical mechanics. Heuristically at low temperature (or at high density), bosons start forming a cycle of different sizes and below a certain critical point, the *infinite* cycles appear in the Feynman–Kac representation. Since this Feynman observation, it is a common wisdom to count this equivalent to (type I) Bose–Einstein condensation [P-O]. For the perfect Bose gas it was proven in [S] and for some weakly interacting Bose gases only recently in [D-M-P, U].

In 1956, Penrose and Onsager [P-O] introduced, via a reduced density matrix, the concept of the off-diagonal-long-range-order (ODLRO) to give an alternative description of Bose–Einstein condensation in ideal or interacting Bose gases. They worked out convincing arguments that existence of ODLRO is equivalent to Bose–Einstein condensation. In fact the ODLRO measures in boson systems a correlation between two infinitely distant points. Recently in papers [U, U2] a contact between the ODLRO and the existence of long cycles was studied.

1.2. Conceptual and physical problems

Although the ODLRO is usually considered as a criterion of (type I) BEC, it is not evident that the ODLRO is equivalent to generalized BEC. Moreover, it is not evident that the presence of long cycles is equivalent to generalized BEC. Therefore, our paper is motivated by two questions:

Are the different criteria of Bose–Einstein condensation (generalized BEC, long cycles and ODLRO) equivalent for the *perfect* Bose gas in anisotropic (Casimir) boxes (section 2)? Hence, the purpose of section 2 is an extension of known results on long cycles, see Suto [S], Dorlas, Martin and Pulé [D-M-P], Ueltschi [U], to Casimir boxes, where the concept of generalized Bose–Einstein condensation can explicitly be verified.

The second question is how we can classify different types of generalized Bose–Einstein condensation with the help of the concept of cycles and with the help of the ODLRO (sections 3–5)?

The aim of this paper is to relate the van den-Berg-Lewis-Pulé classification of g-BEC (types I, II and III) in anisotropic boxes with a hierarchy of long cycles and with the corresponding hierarchy of the ODLRO. More precisely, we would like to know what the scale is of different sizes of the long cycles (macroscopic or not) and correlations (the coherence length) of the condensate?

Our arguments are based on the *scaling approach*. To this end, we propose a scaling formulation for the condensate density and for the notion of long cycles (section 3), as well as for the reduced density matrix and the ODLRO (section 5).

Since the 1970s [dG] scaling concepts were also used in polymer physics. In the present paper we adapt this scaling approach to the cycles (and the two-point correlation function) in both cases because there is a deep analogy between cycle representation of boson systems and polymers [C-S]. Below we give a mathematical presentation of this scaling concept concerning Bose–Einstein condensation of the Ideal Bose gas in Casimir boxes.

This concept is relevant in the physics of quantum coherent states, since it relates the box geometry constraints to the coherence shape of condensate clouds and to the 'geometry' of the boson cycles (polymers' shape). Heuristically, there is a scaling relation between the *coherence length* r and the size of long cycles involving n bosons, which is given by $r = \lambda_\beta n^{1/2}$, provided that the correlation function has the form (5.3). Here $\lambda_\beta = \hbar\sqrt{2\pi\beta/m}$ is the thermal de Broglie length. This relation is analogous to the scaling law for the *ideal polymer chain* [dG], where the size of the chain R is proportional to the number of monomers N: $R = l_0 N^{1/2}$, where l_0 is the effective length of a monomer.

1.3. Results

This paper contains two kind of results interesting for physical properties of the Bose gas in Casimir boxes: $\Lambda = L_1 \times L_2 \times L_3$, with $L_\nu = V^{\alpha_\nu}$, $\nu = 1, 2, 3$, where $\alpha_1 + \alpha_2 + \alpha_3 = 1$. These results establish a contact between generalized van den Berg-Lewis-Pulé condensate and the experimental data concerning the BEC fragmentation [M-H-U-B].

The first result states that the number of particles N_0 in the condensate for a finite-size system (N particles) is (see section 3.2):

$$N_0 = n_1 + n_2 + \cdots + n_M,$$

where n_i are the numbers of particles in the condensate states. Here $n_i = O(N)$ and $M = O(1)$, if $\alpha_1 \leqslant 1/2$ (see theorems 4.1 and 4.2) or $n_i = O(N^\delta)$ and $M = O(N^{1-\delta})$ (such that $N_0 = O(N)$), $\delta = 2(1 - \alpha_1) < 1$, if $\alpha_1 > 1/2$ (see theorem 4.3). This result has a direct relation to the fragmentation theory of Bose–Einstein condensation [M-H-U-B]. The second part of this result (see section 4) is that the order of the size of long cycles is *macroscopic* (i.e., of the order $O(N)$), if we have generalized BEC of type I or II (when $M = O(N)$) and that the order of the size of the long cycles is *microscopic* (of the order $O(N^\delta)$) for generalized BEC of type III, see theorems 4.1–4.3.

The second result is that by virtue of theorem 5.1 the two-point correlation functions (at different scales) have the form

$$\begin{aligned}
\lim_{V\uparrow\infty} \sigma_\Lambda(x - x') &= \rho_0(\beta), \qquad \text{for} \quad \|x - x'\| = O(V^{\alpha_1}), \quad \text{if} \quad \alpha_1 < 1/2,\\
&= \sum_{n_1\in\mathbb{Z}^1} \frac{\cos(2\pi n y)}{\pi\lambda_\beta^2 n^2 + B}, \qquad \text{for} \quad \|x - x'\| = yV^{\alpha_1}, \quad \text{if} \quad \alpha_1 = 1/2,\\
&= \rho_0(\beta)\,\mathrm{e}^{-2y\sqrt{\pi C}/\lambda_\beta}, \qquad \text{for} \quad \|x - x'\| = yV^{\delta/2}, \quad \text{if} \quad \alpha_1 > 1/2,
\end{aligned}$$

where $\rho_0(\beta)$ is the particle density in the condensate, B and C are two positive constants respectively given by (2.8) and (2.9). These forms imply that the order of the *condensate coherence length* coincides with the size of the box in the case of g-BEC of type I or II (i.e., long cycles of *macroscopic* size), but it is smaller if we have generalized BEC of type III (i.e., long cycles of *microscopic* size). The last case shows decreasing of the coherence length for the elongated condensate. Note that existence of this phenomenon is indicated in the physical literature [P-S-W].

To make a contact with experiments involving cold atoms confined in magnetic traps one should extend these results to the Bose gas in a weak harmonic potential. So, an important task is the theoretical and experimental study of very anisotropic cases (such as quasi-2D or quasi-1D systems) to understand the coherence properties of condensate, which mimics the Casimir case.

2. BEC in Casimir boxes

In this section we give a short review of the different concepts concerning Bose–Einstein condensation, such as occupation number, generalized BEC (g-BEC), cycles, long cycles, reduce density matrix, off-diagonal-long-range-order (ODLRO) and the link between the different criteria of BEC (g-BEC, long cycles and ODLRO).

2.1. About the concept of g-BEC condensation and the ODLRO

2.1.1. Basic notions. Let us consider the grand-canonical (β, μ) *perfect* Bose gas (PBG) in Casimir boxes $\Lambda = L_1 \times L_2 \times L_3 \in \mathbb{R}^{d=3}$, of volume $|\Lambda| = V$ with sides of length $L_\nu = V^{\alpha_\nu}$, $\nu = 1, 2, 3$, where $\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 > 0, \alpha_1 + \alpha_2 + \alpha_3 = 1$. For the single-particle Hamiltonian $H_\Lambda^{(N=1)} = T_\Lambda^{(1)} := -(\hbar^2/2m)\Delta$ with periodic boundary conditions, we get the dual vector spaces Λ^* defined by

$$\Lambda^* = \left\{ k \in \mathbb{R}^3 : k = \left(\frac{2\pi n_1}{V^{\alpha_1}}, \frac{2\pi n_2}{V^{\alpha_2}}, \frac{2\pi n_3}{V^{\alpha_3}} \right); n_\nu \in \mathbb{Z}^1 \right\}. \tag{2.1}$$

In the grand-canonical ensemble, the mean values of the k-mode particle densities $\{\rho_\Lambda(k)\}_{k\in\Lambda^*}$ are

$$\rho_\Lambda(k) := \frac{1}{V}\langle N_\Lambda(k)\rangle_\Lambda(\beta, \mu) = \frac{1}{V}\frac{1}{e^{\beta(\epsilon_\Lambda(k)-\mu)} - 1}, \tag{2.2}$$

where $\epsilon_\Lambda(k) = \hbar^2 k^2/2m$, $k \in \Lambda^*$ is the eigenvalue of the Laplacian with periodic boundary conditions, $\langle N_\Lambda(k)\rangle_\Lambda(\beta, \mu)$ is the Gibbs expectation of the particle's number operator $N_\Lambda(k)$ in the mode k. The total density of particles is $\rho_\Lambda(\beta, \mu) := \sum_{k\in\Lambda^*} \rho_\Lambda(k)$.

Let us recall that the eigenfunctions $\left\{\Psi_{\Lambda,k}^{(N=1)}(x)\right\}_{k\in\Lambda^*}$ of the single-particle Hamiltonian $T_\Lambda^{(1)}$ are

$$\Psi_{\Lambda,k}^{(1)}(x) = \frac{1}{\sqrt{V}} e^{ik.x}. \tag{2.3}$$

2.1.2. London scaling approach and g-BEC. In fact it was London [L] who for the first time used implicitly the *scaling* approach to solve the controversy between the Uhlenbeck's mathematical arguments against condensation of the perfect Bose gas for height densities and Einstein's intuitive reasoning in favor of this phenomenon. His line of reasoning was based on the following observations: since the explicit formula for the total particle density $\rho_\Lambda(\beta, \mu)$ in the box Λ is known only in the grand-canonical ensemble (β, μ), to ensure fixed density ρ in this box one has first to solve the equation $\rho = \rho_\Lambda(\beta, \mu)$ which determines the corresponding value of the chemical potential $\overline{\mu}_\Lambda := \overline{\mu}_\Lambda(\beta, \rho)$.

Then the van den Berg–Lewis–Pulé formulation of the g-BEC concept in Casimir boxes has the following form.

Definition 2.1. *We say that the grand-canonical PBG manifests g-BEC for a fixed total density of particle* ρ*, if one has:*

$$\rho_0(\beta, \rho) := \lim_{\epsilon\downarrow 0} \lim_{V\uparrow\infty} \sum_{\{k\in\Lambda^*:\|k\|\leqslant\epsilon\}} \rho_\Lambda(k) > 0, \tag{2.4}$$

where $\rho_\Lambda(k)$ *are defined by* (2.2) *for* $\mu = \overline{\mu}_\Lambda(\beta, \rho)$.

This motivates the following classification:

Definition 2.2.

- *One gets g-BEC of type I if a finite number of the single-particle states are macroscopically occupied.*
- *There is g-BEC of type II is an infinite (countable) number of the single-particle states are macroscopically occupied.*
- *The g-BEC is called type III if none of the single-particle states is macroscopically occupied, but* $\rho_0(\beta, \rho) > 0$.

As usually one introduces for the PBG the critical density, $\rho_c(\beta)$ defined by:

$$\rho_c(\beta) := \sup_{\mu<0} \lim_{V\uparrow\infty} \rho_\Lambda(\beta, \mu) = g_{3/2}(1)\big/\lambda_\beta^3, \tag{2.5}$$

where $g_s(z) := \sum_{j=1}^{\infty} z^j/j^s$ is related to the Riemann zeta-function $\zeta(s) := g_s(1)$. Here $\lambda_\beta = \hbar\sqrt{2\pi\beta/m}$ is the thermal de Broglie length. Note that the critical density does not depend on $\alpha_1, \alpha_2, \alpha_3$, i.e. on the geometry of the boxes. However it is shown in [vdB] that the second critical density noted $\rho_m(\beta)$ to have eventually macroscopic occupation of states could be different from $\rho_c(\beta)$ and depend on the geometry (see perspectives in section 6), but for Casimir boxes these two critical densities are equal.

The following proposition is due to [vdB-L-P]:

Proposition 2.1. *For particle densities* $\rho < \rho_c(\beta)$ *there is no g-BEC of the PBG in Casimir boxes and the chemical potential* $\mu = \overline{\mu}(\beta, \rho)$ *is a unique solution of the equation:*

$$\rho = g_{3/2}(e^{\beta\mu})\big/\lambda_\beta^3. \tag{2.6}$$

Let $1/2 > \alpha_1$*, then for a fixed particle density* $\rho > \rho_c(\beta)$ *the chemical potential* $\overline{\mu}_\Lambda = -A/\beta V + o(1/V)$*, with* $A > 0$ *and there is g-BEC of type I in the single zero mode* $k = 0$*. Here* A *is a solution of the equation:*

$$\rho - \rho_c(\beta) = \frac{1}{A}. \tag{2.7}$$

If $1/2 = \alpha_1$*, then the chemical potential* $\overline{\mu}_\Lambda = -B/\beta V + o(1/V)$*, with* $B > 0$ *and one gets g-BEC of type II in the infinite number of modes:*

$$\lim_{V\uparrow\infty} \rho_\Lambda(k) = \frac{1}{B + \pi\lambda_\beta^2 n_1^2}, \qquad \text{for} \quad k = (2\pi n_1/V^{\alpha_1}, 0, 0), \quad n_1 \in \mathbb{Z}^1,$$

$$= 0, \qquad \text{for} \quad k \neq (2\pi n_1/V^{\alpha_1}, 0, 0), \quad n_1 \in \mathbb{Z}^1.$$

Here B *is a solution of the equation:*

$$\rho - \rho_c(\beta) = \sum_{n_1\in\mathbb{Z}^1} \frac{1}{B + \pi\lambda_\beta^2 n_1^2}. \tag{2.8}$$

If $1/2 < \alpha_1$, *then the chemical potential* $\overline{\mu}_\Lambda = -C/\beta V^\delta + o(1/V^\delta)$, *with* $\delta = 2(1-\alpha_1)$ *and* $C > 0$. *The corresponding g-BEC is of type III: for all* $k \in \Lambda^*$ *we have* $\lim_{V\uparrow\infty} \rho_\Lambda(k) = 0$, *although* $\rho_0(\beta,\rho) > 0$ *and C is a solution of the equation:*

$$\rho - \rho_c(\beta) = \frac{\sqrt{\pi}}{\lambda_\beta C^{1/2}}. \tag{2.9}$$

2.1.3. Reduced density matrix and ODLRO.

Definition 2.3. *We say that the PBG manifests for a fixed total density of particles* ρ *an ODLRO, if one gets a nontrivial limit:*

$$\sigma(\beta,\rho) := \lim_{\|x-x'\|\uparrow\infty} \lim_{V\uparrow\infty} \sigma_\Lambda(\beta,\rho;x,x') > 0, \tag{2.10}$$

where $\sigma(\beta,\rho;x,x') := \lim_{V\uparrow\infty} \sigma_\Lambda(\beta,\rho;x,x')$ *is the two-point correlation function, defined by*

$$\sigma_\Lambda(\beta,\rho;x,x') := \sum_{k\in\Lambda^*} \langle N_\Lambda(k)\rangle_\Lambda(\beta,\rho)\Psi^{(1)}_{\Lambda,k}(x)\Psi^{(1)}_{\Lambda,k}(x')^* = \sum_{k\in\Lambda^*} \rho_\Lambda(k)\,\mathrm{e}^{\mathrm{i}k.(x-x')}, \tag{2.11}$$

for periodic boundary conditions. Here $\rho_\Lambda(k)$ *is defined by (2.2).*

Recall that according to definition 2.1 for a non-zero BEC of types I and II we obtain in the thermodynamical limit a nontrivial particle density in the mode null, i.e. particle density with de Broglie wavelength equal to infinity. This allows a *communication* in the condensate through the whole space (in $\mathbb{R}^3$). This fact is less evident for g-BEC of type III, but some heuristic arguments show that g-BEC and the ODLRO are equivalent for the PBG in Casimir boxes:

Theorem 2.1. *Consider the PBG in Casimir boxes* Λ. *Then,*

$$\begin{aligned} \sigma(\beta,\rho) &= 0, \qquad \text{for} \quad \rho < \rho_c, \\ &= \rho - \rho_c(\beta), \qquad \text{for} \quad \rho > \rho_c, \end{aligned} \tag{2.12}$$

i.e., the ODLRO is non-zero if and only if there is g-BEC.

Proof. We split the correlation function into two parts, the first one is the correlation due to the (future) condensate and the second one corresponds to the particles outside the condensate:

$$\begin{aligned} \sigma(\beta,\rho;x,x') &:= \lim_{\epsilon\downarrow 0}\lim_{V\uparrow\infty} \sum_{k\in\{\Lambda^*:\|k\|\leqslant\epsilon\}} \rho_\Lambda(k)\,\mathrm{e}^{\mathrm{i}k.(x-x')} + \lim_{\epsilon\downarrow 0}\lim_{V\uparrow\infty} \sum_{k\in\{\Lambda^*:\|k\|>\epsilon\}} \rho_\Lambda(k)\,\mathrm{e}^{\mathrm{i}k.(x-x')} \\ &= \lim_{\epsilon\downarrow 0}\lim_{V\uparrow\infty} \sum_{k\in\{\Lambda^*:\|k\|\leqslant\epsilon\}} \rho_\Lambda(k) + \frac{1}{(2\pi)^3}\int_{k\in\mathbb{R}^3} \mathrm{d}k \frac{\mathrm{e}^{\mathrm{i}k.(x-x')}}{\mathrm{e}^{\beta(\hbar^2k^2/2m-\overline{\mu}(\beta,\rho))}-1}, \end{aligned}$$

since $\forall\, k \in \Lambda^* : \|k\| \leqslant \epsilon$, $k.(x-x') \to 0$ when $\epsilon \to 0$. Here $\overline{\mu}(\beta,\rho) := \lim_{V\uparrow\infty} \overline{\mu}_\Lambda(\beta,\rho)$.

Then by virtue of proposition 2.1 and by the Riemann–Lebesgue theorem we obtain the result. □

Since Penrose and Onsager [P-O], the ODLRO has been known as the most relevant criterion of condensation because it is valid with or without interactions between particles. Here we established that ODLRO is not equivalent to the usual criterion of BEC (macroscopic occupation of the ground state) but to generalized BEC. Thus for the PBG a *true* criterion of condensation is generalized BEC. It is natural to suppose (although difficult to check) the same for interacting Bose gases.

2.2. Feynman theory of cycles and existence of long cycles

2.2.1. Feynman concept of cycles. Here we recall the Feynman concept of cycles introduced in 1953 [F]. It is related to the Bose statistic and the Feynman–Kac representation for partition functions.

Recall that for the PBG in boxes $\Lambda = L_1 \times L_2 \times L_3 \subset \mathbb{R}^3$ with *periodic* boundary conditions (i.e., dual vector spaces $\Lambda^* := \{k \in \mathbb{R}^3 : k = (2\pi n_1/L_1, 2\pi n_2/L_2, 2\pi n_3/L_3), n_\nu \in \mathbb{Z}^1\}$, see (2.1)), the grand-canonical pressure has the form

$$p_\Lambda(\beta,\mu) = \frac{1}{\beta|\Lambda|}\log\left(\Xi^0_\Lambda(\beta,\mu)\right) = \frac{1}{\beta|\Lambda|}\sum_{k\in\Lambda^*}\ln[(1-\mathrm{e}^{-\beta(\epsilon_\Lambda(k)-\mu)})^{-1}]$$
$$= \frac{1}{\beta|\Lambda|}\sum_{j=1}^{\infty}\frac{1}{j}\,\mathrm{e}^{j\beta\mu}\,\mathrm{Tr}_{\mathcal{H}^{(1)}_\Lambda}\left(\mathrm{e}^{-j\beta T^{(1)}_\Lambda}\right), \tag{2.13}$$

where $\Xi^0_\Lambda(\beta,\mu)$ is the PBG grand-canonical partition function and we used $\mathrm{Tr}_{\mathcal{H}_{\Lambda,1}}\left(\mathrm{e}^{-j\beta T^{(1)}_\Lambda}\right) = \sum_{\mathbf{k}\in\Lambda^*}\mathrm{e}^{-j\beta\epsilon_\Lambda(k)}$.

The Feynman–Kac representation naturally appears from (2.13) if we consider the representation of the trace of the *Gibbs semigroup*: $\left\{\mathrm{e}^{-\beta T^{(1)}_\Lambda}\right\}_{\beta>0}$, via its kernel $K_\Lambda(\beta; x, x') = \left(\mathrm{e}^{-\beta T^{(1)}_\Lambda}\right)(x, x')$:

$$\mathrm{Tr}_{\mathcal{H}^{(1)}_\Lambda}\left(\mathrm{e}^{-\beta T^{(1)}_\Lambda}\right) = \int_\Lambda \mathrm{d}x\,K_\Lambda(\beta; x, x). \tag{2.14}$$

It is known [G] that the kernel $K_\Lambda(\beta; x, x')$ can be represented as a path integral over Wiener trajectories starting at point x and finishing at point x'. Thus $K_\Lambda(\beta; x, x)$ can be represented as a Wiener integral over closed trajectories (loops) starting and finishing at the same point. The order of the size of the trajectories coincides with the size of the quantum fluctuations λ_β, known as the thermal de Broglie length [M].

By virtue of the Gibbs semigroup properties and by expressions (2.13) and (2.14), we obtain

$$p_\Lambda(\beta,\mu) = \frac{1}{\beta|\Lambda|}\sum_{j=1}^{\infty}\frac{1}{j}\,\mathrm{e}^{j\beta\mu}\int_\Lambda \mathrm{d}x_1\int_\Lambda \mathrm{d}x_2\ldots$$
$$\times\int_\Lambda \mathrm{d}x_j\,K_\Lambda(\beta; x_1, x_2)K_\Lambda(\beta; x_2, x_3)\ldots K_\Lambda(\beta; x_j, x_1). \tag{2.15}$$

In (2.15) each integral over $(x_1,\ldots,x_j) \in \Lambda^j$ correspond to the impact of Wiener's loops of the length $j\lambda_\beta$ [G, M].

Note that the total density of particles in the grand-canonical ensemble is given by

$$\rho_\Lambda(\beta,\mu) := \frac{\langle N_\Lambda\rangle_\Lambda(\beta,\mu)}{V} = \partial_\mu p_\Lambda(\beta,\mu) = \frac{1}{V}\sum_{j=1}^{\infty}\mathrm{e}^{j\beta\mu}\,\mathrm{Tr}_{\mathcal{H}^{(1)}_\Lambda}\,\mathrm{e}^{-j\beta T^{(1)}_\Lambda}. \tag{2.16}$$

One can use this representation to identify the *repartition* of the total density (2.16) over densities of particles involved into loops of the length $j\lambda_\beta$:

$$\rho_{\Lambda,j}(\beta,\mu) := \frac{1}{V}\,\mathrm{e}^{j\beta\mu}\,\mathrm{Tr}_{\mathcal{H}^{(1)}_\Lambda}\,\mathrm{e}^{-j\beta T^{(1)}_\Lambda}. \tag{2.17}$$

The j-loop particle density (2.17) and the representation (2.15) are the key notions for the concept of the *short*/*long* cycles. Indeed, after the thermodynamical limit one can obtain loops of *finite* sizes or *infinite* sizes, i.e. one can relate the BEC to the appearance of loops of infinite size as an explanation of the long-range order and the macroscopic size of the quantum fluctuations [F, P-O, U]. The mathematical basis of the Feynman cycles approach is related to the boson permutation group (see [D-M-P, U, S, M]).

2.2.2. BEC and the concept of short/long cycles.

Definition 2.4. *We say that the representation (2.16) for the grand-canonical PBG contains only short cycles if*

$$\rho_{\text{short}}(\beta,\mu) := \lim_{M\to\infty}\left\{\lim_{V\uparrow\infty}\sum_{j=1}^{M}\rho_{\Lambda,j}(\beta,\mu)\right\} = \rho(\beta,\mu) := \lim_{V\uparrow\infty}\rho_\Lambda(\beta,\mu), \tag{2.18}$$

i.e. it coincides with the total particle density. Since in general the limits in (2.18) are not interchangeable, we say that for a given total particle density ρ the representation (2.16) contains a macroscopic number of long cycles, if $\rho > \rho_{\text{short}}(\beta,\overline{\mu}(\beta,\rho))$, or equivalently if

$$\rho_{\text{long}}(\beta,\rho) := \lim_{M\to\infty}\left\{\lim_{V\uparrow\infty}\sum_{j=M+1}^{\infty}\rho_{\Lambda,j}(\beta,\overline{\mu}_\Lambda(\beta,\rho))\right\} > 0. \tag{2.19}$$

Since Feynman [F] the presence of the non-zero density of the long cycles has usually been connected with the existence of zero-mode BEC, but a rigorous proof of this conjecture for a certain class of models has been obtained only recently. There we noted that even for the PBG type I BEC the mathematical proof of this connection was not straightforward and appealed to a non-trivial analysis (see [S, D-M-P, U]).

Theorem 2.2. *Let us consider a PBG in Casimir boxes Λ. Then we have*

$$\begin{aligned}\rho_{\text{long}}(\beta,\rho) &= 0, \qquad \textit{if} \quad \rho < \rho_c(\beta),\\ &= \rho - \rho_c(\beta), \qquad \textit{if} \quad \rho > \rho_c(\beta),\end{aligned} \tag{2.20}$$

where $\rho_c(\beta)$ is the critical density defined by (2.5).

Proof. By virtue of (2.17) and (2.18) we have

$$\begin{aligned}\rho_{\text{short}}(\beta,\mu) &= \lim_{M\to\infty}\left\{\lim_{V\uparrow\infty}\sum_{j=1}^{M}\mathrm{e}^{j\beta\mu}\prod_{\nu=1}^{3}\frac{1}{V^{\alpha_\nu}}\sum_{n_\nu\in\mathbb{Z}^1}\mathrm{e}^{-j\beta\pi\lambda_\beta^2(n_\nu/V^{\alpha_\nu})^2}\right\}\\ &= \sum_{j=1}^{\infty}\mathrm{e}^{j\beta\mu}\prod_{\nu=1}^{3}\int_{\mathbb{R}}\mathrm{d}\xi_\nu\,\mathrm{e}^{-j\beta\pi\lambda_\beta^2\xi_\nu^2} = \sum_{j=1}^{\infty}\frac{\mathrm{e}^{\beta\mu}}{\lambda_\beta^3 j^{3/2}} =: \frac{g_{3/2}(\mathrm{e}^{\beta\mu})}{\lambda_\beta^3},\end{aligned}$$

c.f. definition of $g_s(z)$. By virtue of proposition 2.1, if $\rho < \rho_c(\beta)$ we have $\rho_{\text{short}}(\beta,\rho) = \rho$ and if $\rho > \rho_c(\beta)$ we have $\rho_{\text{short}}(\beta,\rho) = \rho_c(\beta)$. So, by definition 2.4 we conclude the proof. □

Intuitively the size of long cycles for the usual BEC (ground-state macroscopic occupation) is of the order of the total particle number ($j = O(N)$). What happens if the condensate is fragmented, i.e. of type II or III? Below we apply a scaling approach to study these cases and we prove in theorem 4.3 that the order of the size of long cycles is smaller than the number of particles ($O(N^\delta)$, $\delta < 1$).

3. Generalized BEC concept: revisited

In this section we propose a modification of the concept of generalized BEC, which we call *scaled* BEC (s-BEC). This implies the corresponding modification of the concept for cycles (*scaled* short/long cycles denoted s-short/s-long cycles) via similar scaling arguments. Moreover, we introduce a classification of s-BEC (types I, II, III) and the *hierarchy* of the s-long cycles, which distinguishes long-microscopic/macroscopic cycles.

3.1. Generalized condensation and scaled condensation

The original van den Berg–Lewis–Pulé concept of g-BEC [vdB-L-P] was not explicitly addressed to detect a *fine* structure of the condensate: *a priori* it does not allow the distinction to be made between generalized BEC of *types* I, II or III. In fact one can do this analysis, as was done for the first time in [vdB-L] for the case of Casimir boxes.

To make this facet more evident we introduce in this paper a *new* definition of *generalized* BEC, which we call *scaled* BEC (s-BEC). Take for simplicity the PBG in Casimir boxes Λ with *periodic* boundary conditions, i.e. with the dual vector spaces Λ^* defined by (2.1), and with the k-modes mean particle densities $\{\rho_\Lambda(k)\}_{k\in\Lambda}$ defined by (2.2).

Definition 3.1. *We say that for a fixed total density ρ the (perfect) Bose gas manifests s-BEC in boxes Λ, if there exists a positive decreasing function $\eta : V \mapsto \mathbb{R}_+$, such that* $\lim_{V\uparrow\infty}\eta(V) = 0$ *and we have*

$$\rho_\eta(\beta,\rho) := \liminf_{V\uparrow\infty} \sum_{\{k\in\Lambda^*:\|k\|\leqslant\eta(V)\}} \rho_\Lambda(k) > 0. \tag{3.1}$$

Remark 3.1. Recall that the van den Berg–Lewis–Pulé definition of g-BEC is formulated as

$$\rho_0(\beta,\rho) := \lim_{\epsilon\downarrow 0}\lim_{V\uparrow\infty} \sum_{\{k\in\Lambda^*:\|k\|\leqslant\epsilon\}} \rho_\Lambda(k) > 0. \tag{3.2}$$

Hence, the two definitions 2.1 and 3.1 are evidently not equivalent. Moreover, we show that our definition 3.1 also allows us to connect a fine mode structure of the condensate of types I, II or III with the long-cycles hierarchy, and to show that there is a relation between the structure of the condensate and the size of cycles.

The following statement is an evident consequence of definitions 2.1 and 3.1.

Lemma 3.1. *For any function $\eta(V)$ we have*

$$0 \leqslant \rho_\eta(\beta,\rho) \leqslant \rho_0(\beta,\rho). \tag{3.3}$$

This lemma means that s-BEC implies g-BEC (i.e., if there is no g-BEC, then there is no s-BEC).

A simple example of application of the s-BEC approach is the possibility of distinguishing type I, II or III condensations of the PBG in Casimir boxes Λ with *periodic* boundary conditions.

Proposition 3.1. *The rate $\eta(V) = O(1/V^{1/2})$ is an important threshold to refine a discrimination between different types of g-BEC. If for example, one takes $\eta_\delta(V) = 2\pi/V^{(1/2-\delta)}$ such that $\delta > 0$, then we obtain*

$$\rho_{\eta_\delta}(\beta,\rho) = \rho_0(\beta,\rho), \qquad \text{for} \quad \alpha_1 \leqslant 1/2,$$

$$\rho_{\eta_\delta}(\beta,\rho) = 0, \qquad \text{for} \quad \alpha_1 > 1/2+\delta,$$

$$\rho_{\eta_\delta}(\beta,\rho) = \rho_0(\beta,\rho), \qquad \text{for} \quad 1/2+\delta > \alpha_1 > 1/2.$$

On the other hand, for $\alpha_1 = 1/2$ *and* $\eta_\Gamma(V) := 2\pi\Gamma/V^{1/2}$ *one obtains a modification of the density of the type II condensation:*

$$\rho_{\eta_\Gamma}(\beta,\rho) = \sum_{|n_1|\leqslant\Gamma} \frac{1}{\pi\lambda_\beta^2 n_1^2 + B} < \rho_0(\beta,\rho).$$

For $\alpha_1 > 1/2$ *and* $\eta_{\Gamma'}(V) := 2\pi\Gamma'/V^{1-\alpha_1}$ *one obtains a modification of the density of the type III condensation:*

$$\rho_{\eta_{\Gamma'}}(\beta,\rho) = \int_{\mathbb{R}^+} \mathrm{d}\zeta \, \frac{\mathrm{e}^{-C\zeta}}{\zeta^{1/2}\lambda_\beta} \operatorname{erf}(\Gamma'\lambda_\beta\sqrt{\zeta\pi}) < \rho_0(\beta,\rho),$$

where $\operatorname{erf}(\cdot)$ *stands for error function and where* C *is the unique solution of equation (2.9).*

Proof. By virtue of the proof of theorems 4.1–4.3, with the different choices of $\eta(V)$, we can obtain the results. □

Note that the scaling criterion of condensation (definition 3.1) is more adapted for physical description of BEC because a judicious choice of the function $\eta(V)$ may give support for momentum distribution of the condensate. It also might be useful for numerical analysis of experimentally observed *fragmented* condensates [M-H-U-B].

Let N_0 be the number of particles in the condensate. Then we say that there is fragmentation of first type if $N_0 = n_1 + n_2 + \cdots + n_M$ with $n_i = O(N)$ and $M = O(1)$ (where n_i are the numbers of particles in the condensate states), or of the second type if $n_i = O(N^\delta)$ and $M = O(N^{1-\delta})$, $\delta < 1$ (such that $N_0 = O(N)$).

3.2. Scaling approach of the short/long cycles

First we introduce the concepts of *scaled* short/long cycles (s-short/long cycles):

Definition 3.2. *We say that Bose gas manifests s-long cycles, if there exists a positive increasing function of the volume* $\lambda : \mathbb{R}^+ \to \mathbb{N}^+$*, such that* $\lim_{V\uparrow\infty}\lambda(V) = \infty$ *and*

$$\rho_{\mathrm{long},\lambda}(\beta,\rho) := \liminf_{V\uparrow\infty} \sum_{j\geqslant\lambda(V)} \rho_{\Lambda,j}(\beta,\overline{\mu}_\Lambda(\beta,\rho)) > 0, \tag{3.4}$$

where $\rho_{\Lambda,j}(\beta,\mu)$ *is given by (2.17).*

The following statement is an evident consequence of definitions 2.4 and 3.2:

Lemma 3.2. *In the particular case of PBG in Casimir boxes we have*

$$0 \leqslant \rho_{\mathrm{long},\lambda}(\beta,\rho) \leqslant \rho_{\mathrm{long}}(\beta,\rho), \tag{3.5}$$

for any function $\lambda(V)$*. Here* $\rho_{\mathrm{long}}(\beta,\rho)$ *is given by (2.19).*

This lemma implies that the presence of s-long cycles implies the presence of long cycles (cf definition 2.4).

A simple example of the application of the s-long cycles approach is the possibility of distinguishing the type I, II or III condensations of the PBG in Casimir boxes Λ with *periodic* boundary conditions.

Proposition 3.2. *If* $\lambda(V) = V^{\delta}$ *then for* $\rho \geqslant \rho_c(\beta)$ *we obtain*

$$\begin{aligned}
&\rho_{\text{long},\lambda}(\beta,\rho) = 0, && \text{for} \quad \delta > 1,\\
&\rho_{\text{long},\lambda}(\beta,\rho) = \rho_0(\beta,\rho), && \text{for} \quad \alpha_1 \leqslant 1/2, 0 < \delta < 1,\\
&\rho_{\text{long},\lambda}(\beta,\rho) = 0, && \text{for} \quad \alpha_1 > 1/2, 2(1-\alpha_1) < \delta,\\
&\rho_{\text{long},\lambda}(\beta,\rho) = \rho_0(\beta,\rho), && \text{for} \quad \alpha_1 > 1/2, 0 < \delta < 2(1-\alpha_1).
\end{aligned}$$

Proof. Adapting the proof of theorems 4.1–4.3 with different choices of δ for $\lambda(V) = V^{\delta}$ we can obtain the results. □

Definition 3.3. *If* $j : \mathbb{R}^+ \to \mathbb{N}^+$ *is a bounded positive increasing function of the volume, i.e.* $\lim_{V\uparrow\infty} j(V) < \infty$*, then* $\rho_{\Lambda,j(V)}(\beta, \overline{\mu}_\Lambda(\beta,\rho))$ *is the density of particles in the s-short cycles of size* $j(V)$*.*

Definition 3.4. *If* $j : \mathbb{R}^+ \to \mathbb{N}^+$ *is a positive increasing function of the volume such as* $\lim_{V\uparrow\infty} j(V) = \infty$*, then* $\rho_{\Lambda,j(V)}(\beta, \overline{\mu}_\Lambda(\beta,\rho))$ *is the density of particles in the s-long cycles of size* $j(V)$*.*

There is a natural classification of s-long cycles:

- *if* $\lim_{V\uparrow\infty}(j(V)/V) = 0$*, we say that* $\rho_{\Lambda,j(V)}(\beta, \overline{\mu}_\Lambda(\beta,\rho))$ *is the density of particles in the microscopic-long cycles of size* $j(V)$*,*
- *if* $0 < \lim_{V\uparrow\infty}(j(V)/V) < \infty$*, we say that* $\rho_{\Lambda,j(V)}(\beta, \overline{\mu}_\Lambda(\beta,\rho))$ *is the density of particles in the macroscopic cycles of size* $j(V)$*,*
- *if* $\lim_{V\uparrow\infty}(j(V)/V) = \infty$*, we say that* $\rho_{\Lambda,j(V)}(\beta, \overline{\mu}_\Lambda(\beta,\rho))$ *is the density of particles in the large cycles of size* $j(V)$*.*

To clarify definition 3.4, we make the following remark:

Remark 3.2. We say that if $j(V)$ in (2.17) is of the order of V^{α} if $0 < \lim_{V\uparrow\infty}(j(V)/V^{\alpha}) < \infty$, for example $j(V) = xV^{\alpha}$, $x > 0$. If $\alpha < 1$, we are in the first case of the classification in definition 3.4. This case is important because it creates a question: can we have a macroscopic quantity of particles in the microscopic-long-cycles? If $\alpha = 1$ we are in the second case of the classification of s-long-cycles and if $\alpha > 1$ we are in the third case. Of course, one can take above any increasing function $j(V)$ including e.g. $\ln(V)$.

3.3. Hierarchy of s-long cycles

Definitions 3.3, 3.4 and remark 3.2 allow one to give a natural classification of scaled-long (s-long) cycles. We call this classification a *hierarchy* of cycles, ordered according to their size.

In general there are long cycles of any size:

Definition 3.5. *We say that the Bose gas manifests s-long cycles of the order* $\lambda(V)$ *where* $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ *is a positive increasing function of the volume, if there exist two positive real numbers x and y such as the s-long cycle's particle density:*

$$\lim_{V\uparrow\infty} \sum_{j=x\lambda(V)}^{y\lambda(V)} \rho_{\Lambda,j}(\beta, \overline{\mu}_\Lambda(\beta,\rho)) > 0.$$

Then the total density of particles in cycles of size of the order $\lambda(V)$ is:

$$\rho_{\text{long}}(\beta,\rho|\lambda) := \lim_{x\downarrow 0;y\uparrow\infty}\lim_{V\uparrow\infty}\sum_{j=x\lambda(V)}^{y\lambda(V)}\rho_{\Lambda,j}(\beta,\overline{\mu}_{\Lambda}(\beta,\rho)).$$

We introduce a classification of s-long cycles:

Definition 3.6.

- *We say that the Bose gas manifests macroscopic cycles, if the Bose gas manifests s-long cycles of the order V.*
- *We say that the Bose gas manifests long-microscopic cycles, if the Bose gas manifests s-long cycles of orders smaller than V.*

In the next section we see that in the particular case of the PBG in Casimir boxes, the classification of s-BEC induces a hierarchy in classification of s-long cycles. By virtue of theorems 4.1–4.3 we obtain the following proposition.

Proposition 3.3. *If x and y are two positive real numbers, then we have*

$$\begin{aligned}\lim_{\Lambda}\sum_{j=xV^{\delta}}^{yV^{\delta}}\rho_{\Lambda,j}(\beta,\overline{\mu}_{\Lambda}(\beta,\rho)) &= (\mathrm{e}^{-xA}-\mathrm{e}^{-yA})\rho_0(\beta,\rho), \quad \text{for} \quad \alpha_1<1/2, \delta=1,\\ &= (\mathrm{e}^{-xB}-\mathrm{e}^{-yB})\rho_0(\beta,\rho), \quad \text{for} \quad \alpha_1=1/2, \delta=1,\\ &= (\mathrm{e}^{-xC}-\mathrm{e}^{-yC})\rho_0(\beta,\rho) \quad \text{for} \quad \alpha_1>1/2, \delta=2(1-\alpha_1),\end{aligned}$$

where A is the unique solution of equation (2.7), B is the unique solution of equation (2.8) and C is the unique solution of equation (2.9).

This proposition gives an illustration of the hierarchy of cycles, in the first case ($\alpha_1<1/2$) and second case ($\alpha_1=1/2$) the Bose gas manifests the presence of macroscopic cycles, and in the third case ($\alpha_1>1/2$) the Bose gas manifests the presence of long-microscopic cycles of the order $V^{2(1-\alpha_1)}$. We are going to discuss in detail the link with the results given in proposition 2.1 in the next section.

4. Does generalized BEC of types I, II, III imply a hierarchy of long cycles?

The aim of this section is to relate a fine structure of g-BEC and s-BEC (type I, II or III) with a hierarchy of long cycles. Recall that we deal with Casimir boxes $\Lambda = V^{\alpha_1}\times V^{\alpha_2}\times V^{\alpha_3}$, $\alpha_1\geqslant\alpha_2\geqslant\alpha_3>0$, $|\Lambda|=V$, $(\alpha_1+\alpha_2+\alpha_3=1)$.

4.1. Generalized BEC in the case: $\alpha_1<1/2$

In this case the geometry is similar to the one for the usual cubic box. Our main result here is that g-BEC of type I implies macroscopic cycles in the fundamental state.

Theorem 4.1. *If one takes Casimir boxes Λ with $1/2>\alpha_1$, then for a fixed density of particles $\rho>\rho_c(\beta)$ the chemical potential is $\overline{\mu}_{\Lambda}:=\overline{\mu}_{\Lambda}(\beta,\rho)=-A/\beta V+o(1/V)$, with $A>0$. This implies s-BEC as well as g-BEC of type I in the zero mode (ground state) together only with macroscopic cycles in this mode. Here A is the unique solution of equation (2.7).*

Proof. Taking into account (2.1) we denote the family of Casimir boxes by Λ_I with the corresponding dual space:

$$\Lambda_I^* := \left\{ k \in \mathbb{R}^3 : k = \left(\frac{2\pi n_1}{V^{\alpha_1}}, \frac{2\pi n_2}{V^{\alpha_2}}, \frac{2\pi n_3}{V^{\alpha_3}} \right); n_\nu \in \mathbb{Z}^1; 1/2 > \alpha_1 \right\}. \tag{4.1}$$

Let $\Lambda_{0,I}^*$ be a subset of Λ_I^* defined by

$$\Lambda_{0,I}^* = \{ k \in \Lambda_I^* : \|k\| \leqslant \eta_I(V) \}, \tag{4.2}$$

where $\eta_I(V) = 1/V$. Then we have $\Lambda_{0,I}^* = \{k = 0\}$.

Let us consider the total density of particles:

$$\rho := \lim_{V \uparrow \infty} \rho_\Lambda(\beta, \overline{\mu}_\Lambda) = \rho_{\text{short}}(\beta, \rho) + \rho_{\text{long}}(\beta, \rho), \tag{4.3}$$

cf (2.18) and (2.19).

We can decompose the density of particles in long cycles into two parts defined by

$$\rho_{\text{long}}(\beta, \rho) := \rho_{\text{long}}(\Lambda_I^* \backslash \Lambda_{0,I}^*) + \rho_{\text{long}}(\Lambda_{0,I}^*), \tag{4.4}$$

where $\rho_{\text{long}}(\Lambda_I^* \backslash \Lambda_{0,I}^*)$ is the limiting density of particles in long cycles outside $\Lambda_{0,I}^*$:

$$\rho_{\text{long}}(\Lambda_I^* \backslash \Lambda_{0,I}^*) := \lim_{M \to \infty} \lim_{V \uparrow \infty} \left(\sum_{k \in \Lambda_I^* \backslash \Lambda_{0,I}^*} \sum_{j=M}^{\infty} \rho_{\Lambda,j}(k) \right), \tag{4.5}$$

where the spectral repartition of particle density in j-cycles is

$$\rho_{\Lambda,j}(k) := \frac{1}{V} \mathrm{e}^{j\beta\overline{\mu}_\Lambda} \mathrm{e}^{-j\beta\epsilon_\Lambda(k)}, \tag{4.6}$$

and $\overline{\mu}_\Lambda := \overline{\mu}_\Lambda(\beta, \mu)$ is the solution of the equation $\rho = \rho_\Lambda(\beta, \mu)$.

First we shall estimate the density of particles in long cycles of $\Lambda_I^* \backslash \Lambda_{0,I}^*$ by (4.5) and asymptotic for $\overline{\mu}_\Lambda$ we obtain

$$\begin{aligned} \rho_{\text{long}}(\Lambda_I^* \backslash \Lambda_{0,I}^*) &= \lim_{M \to \infty} \lim_{V \uparrow \infty} \left(\sum_{k \in \Lambda_I^* \backslash \Lambda_{0,I}^*} \sum_{j=M}^{\infty} \frac{1}{V} \mathrm{e}^{j\beta\overline{\mu}_\Lambda} \mathrm{e}^{-j\beta\epsilon_\Lambda(k)} \right), \\ &= \lim_{M \to \infty} \sum_{j=M}^{\infty} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \mathrm{d}k \, \mathrm{e}^{-j\pi\lambda_\beta^2 k^2}, \\ &= 0. \end{aligned}$$

Consequently there are no long cycles in $\Lambda_I^* \backslash \Lambda_{0,I}^*$ and since our last estimate is valid for any $M \to \infty$ we conclude that there are no s-long cycles in $\Lambda_I^* \backslash \Lambda_{0,I}^*$ (see definition 3.2).

Now let us consider the modes in $\Lambda_{0,I}^*$, we would like to prove that the PBG manifests s-long cycles of the order $O(V)$, i.e. macroscopic cycles (see definition 3.6).

Since $\overline{\mu}_\Lambda = -A/\beta V + o(1/V)$, with $A > 0$ we have

$$\begin{aligned} \rho_{\text{long}}(\Lambda_{0,I}^* | \text{macro}) &:= \lim_{x \downarrow 0; y \uparrow \infty} \lim_{V \uparrow \infty} \sum_{j=xV}^{yV} \frac{1}{V} \mathrm{e}^{j\beta\overline{\mu}_\Lambda} \\ &= \lim_{x \downarrow 0; y \uparrow \infty} \lim_{V \uparrow \infty} \frac{(\mathrm{e}^{-xA+O(1/V)} - \mathrm{e}^{-yA+O(1/V)})}{\mathrm{e}^{-\beta\overline{\mu}_\Lambda} - 1} \\ &= \lim_{V \uparrow \infty} \frac{1}{\mathrm{e}^{-\beta\overline{\mu}_\Lambda} - 1} = \lim_{V \uparrow \infty} \rho_\Lambda(\Lambda_{0,I}^*), \end{aligned} \tag{4.7}$$

where $\rho_\Lambda(\Lambda_{0,I}^*) := \sum_{k \in \Lambda_{0,I}} \rho_\Lambda(k)$ is the density of particles in $\Lambda_{0,I}^*$.

We can easily calculate $\rho_\Lambda(\Lambda^*_{0,I})$:

$$\rho_\Lambda(\Lambda^*_{0,I}) = \frac{1}{V}\frac{1}{\mathrm{e}^{-\beta\overline{\mu}_\Lambda}-1} = \frac{1}{V}\frac{1}{\mathrm{e}^{\beta\frac{A}{\beta V}+o(\frac{1}{V})}-1} = \frac{1}{A} + o\left(\frac{1}{V}\right). \tag{4.8}$$

So by virtue of (4.4), (4.7) and (4.8),

$$\rho_{\mathrm{long}}(\Lambda^*_{0,I}|\mathrm{macro}) = \rho_{\mathrm{long}}(\beta,\rho) = \frac{1}{A}. \tag{4.9}$$

We know by theorem 2.2 that the density of particles in short cycles is equal to the critical density. Consequently by virtue of (4.3) and (4.9) we can conclude the proof of the theorem. □

4.2. *Generalized BEC in the case: $\alpha_1 = 1/2$*

The main result of this subsection is a theorem about g-BEC of type II, which is related to the presence of macroscopic cycles in an infinite (in the thermodynamical limit) number of modes.

Theorem 4.2. *For Casimir boxes with $1/2 = \alpha_1$ and a fixed particle density $\rho > \rho_c(\beta)$ the chemical potential is $\overline{\mu}_\Lambda = -B/\beta V + o(1/V)$, with $B > 0$. This implies s-BEC as well as g-BEC of type II in an infinite (in thermodynamical limit) number of modes and simultaneously macroscopic cycles only in these modes. Here B is a unique solution of equation (2.8).*

Proof. Taking into account (2.1) we denote the family of Casimir boxes by Λ_{II} and the dual space is

$$\Lambda^*_{II} = \left\{k\in\mathbb{R}^3 : k = \left(\frac{2\pi n_1}{V^{\alpha_1}},\frac{2\pi n_2}{V^{\alpha_2}},\frac{2\pi n_3}{V^{\alpha_3}}\right); n_\nu\in\mathbb{Z}^1; 1/2=\alpha_1\right\}, \tag{4.10}$$

Let $\Lambda^*_{0,II,\Gamma}$ be a subset of Λ^*_{II}

$$\Lambda^*_{0,II,\Gamma} = \left\{k\in\Lambda^*_{II} : \|k\| \leqslant \eta^\Gamma_{II}(V)\right\}, \tag{4.11}$$

where

$$\eta^\Gamma_{II}(V) := \frac{2\pi\Gamma}{V^{1/2}}, \Gamma\in\mathbb{N}^*. \tag{4.12}$$

Note that this set contains the whole value of the condensate as well as particles involved in the long cycles for $\Gamma\to\infty$ after the thermodynamical limit.

Again we decompose the density of particles in long cycles into two parts:

$$\rho_{\mathrm{long}}(\beta,\rho) = \lim_{\Gamma\uparrow\infty}\rho_{\mathrm{long}}(\Lambda^*_{II}\backslash\Lambda^*_{0,II,\Gamma}) + \lim_{\Gamma\uparrow\infty}\rho_{\mathrm{long}}(\Lambda^*_{0,II,\Gamma}). \tag{4.13}$$

We take the limit $\Gamma\to\infty$ to have the totality of the condensate in the first part.

The second part of (4.13) is

$$\begin{aligned}\lim_{\Gamma\uparrow\infty}\rho_{\mathrm{long}}(\Lambda^*_{II}\backslash\Lambda^*_{0,II,\Gamma}) &= \lim_{\Gamma\uparrow\infty}\lim_{M\uparrow\infty}\lim_{\Lambda}\left(\sum_{j=M}^{\infty}\frac{1}{V}\mathrm{e}^{-j\beta C/V}\sum_{\|k\|>2\pi\Gamma/\sqrt{V}}\mathrm{e}^{-j\beta\epsilon_\Lambda(k)}\right)\\ &= \lim_{\Gamma\uparrow\infty}\lim_{M\uparrow\infty}\lim_{\Lambda}\left(\sum_{j=M}^{\infty}\frac{1}{V}\mathrm{e}^{-j\beta C/V}\sum_{1/V^{1/2-\epsilon}>\|k\|>2\pi\Gamma/\sqrt{V}}\mathrm{e}^{-j\beta\epsilon_\Lambda(k)}\right)\\ &\quad+\lim_{\Gamma\uparrow\infty}\lim_{M\uparrow\infty}\lim_{\Lambda}\left(\sum_{j=M}^{\infty}\frac{1}{V}\mathrm{e}^{-j\beta C/V}\sum_{\|k\|>1/V^{1/2-\epsilon}}\mathrm{e}^{-j\beta\epsilon_\Lambda(k)}\right),\end{aligned} \tag{4.14}$$

with $1/2 - \epsilon > \alpha_2$. Then we calculate the second term of (4.14):

$$\lim_{\Gamma\uparrow\infty}\lim_{M\uparrow\infty}\lim_{\Lambda}\left(\sum_{j=M}^{\infty}\frac{1}{V}\,\mathrm{e}^{-j\beta C/V}\sum_{\|k\|>1/V^{1/2-\epsilon}}\mathrm{e}^{-j\beta\epsilon_\Lambda(k)}\right)=\lim_{M\uparrow\infty}\sum_{j=M}^{\infty}\frac{1}{j^{3/2}\lambda_\beta^3}=0.$$

The first term of (4.14) has an upper bound:

$$\lim_{\Gamma\uparrow\infty}\lim_{\Lambda}\left(\sum_{j=1}^{\infty}\frac{1}{V}\,\mathrm{e}^{-j\beta C/V}\sum_{1/V^{1/2-\epsilon}>\|k\|>2\pi\Gamma/\sqrt{V}}\mathrm{e}^{-j\beta\epsilon_\Lambda(k)}\right)=\lim_{\Gamma\uparrow\infty}\sum_{|n_1|>\Gamma}\frac{1}{\pi\lambda_\beta^2 n_1^2+B}=0,$$

consequently the first term of (4.14) is null so $\lim_{\Gamma\uparrow\infty}\rho_{\text{long}}(\Lambda^*_{II}\backslash\Lambda^*_{0,II,\Gamma})=0$.

Now let us consider the modes in $\Lambda^*_{0,II,\Gamma}$. We would like to apply the same strategy as the proof of theorem 4.1 to prove that the PBG manifests s-long cycles of the order $O(V)$, i.e. macroscopic cycles (see definition 3.6).

Since $\overline{\mu}_\Lambda = -B/\beta V + o(1/V)$, with $B>0$ we have

$$\begin{aligned}\lim_{\Gamma\uparrow\infty}\rho_{\text{long}}(\Lambda^*_{0,II,\Gamma}|\text{macro}) &:= \lim_{\Gamma\uparrow\infty}\lim_{x\downarrow 0;y\uparrow\infty}\lim_{V\uparrow\infty}\frac{1}{V}\sum_{k\in\Lambda^*_{0,II}}\sum_{j=xV}^{yV}\rho_{\Lambda,j}(k)\\ &=\lim_{\Gamma\uparrow\infty}\lim_{x\downarrow 0;y\uparrow\infty}\lim_{V\uparrow\infty}\sum_{k\in\Lambda^*_{0,II,\Gamma}}\frac{(\mathrm{e}^{-xB+O(1/V)}-\mathrm{e}^{-yB+O(1/V)})}{\mathrm{e}^{\beta(\epsilon_\Lambda(k)-\overline{\mu}_\Lambda)}-1}\\ &=\lim_{\Gamma\uparrow\infty}\lim_{V\uparrow\infty}\sum_{k\in\Lambda^*_{0,II,\Gamma}}\frac{1}{\mathrm{e}^{\beta(\epsilon_\Lambda(k)-\overline{\mu}_\Lambda)}-1}\\ &=\lim_{\Gamma\uparrow\infty}\lim_{V\uparrow\infty}\left(\sum_{\mathbf{k}\in\Lambda^*_{0,II,\Gamma}}\rho_\Lambda(\mathbf{k})\right).\end{aligned}\tag{4.15}$$

We can easily calculate

$$\begin{aligned}\lim_{\Gamma\uparrow\infty}\lim_{V\uparrow\infty}\left(\sum_{\mathbf{k}\in\Lambda^*_{0,II,\Gamma}}\rho_\Lambda(\mathbf{k})\right) &= \lim_{\Gamma\uparrow\infty}\lim_{V\uparrow\infty}\sum_{n_1=-\Gamma}^{\Gamma}\frac{1}{V}\frac{1}{\mathrm{e}^{\beta(\pi\lambda_\beta^2 n_1^2/V+B/V+O(1/V))}-1}\\ &=\sum_{n_1\in\mathbb{Z}^1}\frac{1}{B+\pi\lambda_\beta^2 n_1^2}.\end{aligned}\tag{4.16}$$

So by virtue of (4.13), (4.15) and (4.16),

$$\lim_{\Gamma\uparrow\infty}\rho_{\text{long}}(\Lambda^*_{0,II,\Gamma}|\text{macro})=\rho_{\text{long}}(\beta,\rho)=\sum_{n_1\in\mathbb{Z}^1}\frac{1}{B+\pi\lambda_\beta^2 n_1^2}.\tag{4.17}$$

We know by theorem 2.2 that the density of particles in short cycles is equal to the critical density. Consequently by virtue of (4.3) and (4.17) we can conclude the proof of the theorem. □

4.3. Generalized BEC in the case: $\alpha_1 > 1/2$

Our main result is the theorem about g-BEC of type III due to the presence of *long-microscopic* cycles in infinite (in the thermodynamical limit) number of modes.

Theorem 4.3. *If one takes the Casimir boxes $\Lambda = V^{\alpha_1}\times V^{\alpha_2}\times V^{\alpha_3}$ with $1/2>\alpha_1$, then for a fixed density of particles $\rho>\rho_c(\beta)$ the chemical potential is $\overline{\mu}_\Lambda(\beta,\rho)=-C/\beta V^\delta + o(1/V^\delta)$,* *with $\delta = 2(1-\alpha_1)$ and $C > 0$. This implies s-BEC as well as g-BEC of type III in infinite (in the thermodynamical limit) number of modes, together with long-microscopic cycles of the order V^{δ}, but only in these modes. Here C is a unique solution of equation (2.9).*

Proof. Taking into account (2.1) we denote the family of Casimir boxes Λ_{III} and the dual spaces are defined by

$$\Lambda^*_{III} = \left\{k \in \mathbb{R}^3 : k = \left(\frac{2\pi n_1}{V^{\alpha_1}}, \frac{2\pi n_2}{V^{\alpha_2}}, \frac{2\pi n_3}{V^{\alpha_3}}\right); n_i \in \mathbb{Z}^1; \alpha_1 > 1/2\right\}. \quad (4.18)$$

Let $\Lambda^*_{0,III,\Gamma}$ be a subset of Λ^*_{III}:

$$\Lambda^*_{0,III,\Gamma} = \left\{k \in \Lambda^*_{III} : \|k\| \leqslant \eta^{\Gamma}_{III}(V)\right\}, \quad (4.19)$$

where

$$\eta^{\Gamma}_{III}(V) := \frac{2\pi\Gamma}{V^{\delta/2}}, \Gamma \in \mathbb{N}^*, \quad (4.20)$$

where $\delta = 2(1-\alpha_1) < 1$.

We show that $\Lambda^*_{0,III,\Gamma}$ contains the whole value of the condensate as well as the particles involved into long cycles in the limit $\Gamma \to \infty$.

Before presenting formal arguments, let us make a remark about the *qualitative* difference between the cases $\alpha_1 > 1/2$ and $\alpha_1 \leqslant 1/2$. With the definition of $\eta^{\Gamma}_{III}(V)$ (4.20), we see that the number of states in $\Lambda^*_{0,III,\Gamma}$ is of the order $O(V^{2\alpha_1-1})$ that goes to infinity, with increasing volume, there are many more states in the condensate than in $\Lambda^*_{0,II,\Gamma}$ (defined by (4.11)). Heuristically one can say that $\Lambda^*_{0,III,\Gamma}$ contains long cycles of sizes of the order $O(V^{\delta})$ in a number of modes of the order $O(V^{2\alpha_1-1})$. Thus the number of particles in these s-long cycles is of the order $O(V^{\delta})O(V^{2\alpha_1-1}) = O(V)$, which is macroscopic. For this reason there is a macroscopic condensate (the order of the number of particles is $O(V)$) because there is an accumulation of microscopic condensates (the order of the number of particles is $O(V^{2\alpha_1-1})$ which is smaller than $O(V)$) as well as an accumulation of long-microscopic cycles in $\Lambda^*_{0,III,\Gamma}$ at each mode of the condensate.

Again we decompose the density of particles in long cycles into two parts:

$$\rho_{\text{long}}(\beta,\rho) = \lim_{\Gamma\uparrow\infty} \rho_{\text{long}}(\Lambda^*_{III}\backslash\Lambda^*_{0,III,\Gamma}) + \lim_{\Gamma\uparrow\infty} \rho_{\text{long}}(\Lambda^*_{0,III,\Gamma}). \quad (4.21)$$

By the same argument as in the proof of theorem 4.2, one finds that the first term of (4.21) is null.

Now let us consider the modes in $\Lambda^*_{0,III,\Gamma}$. In this case, we would like to prove that the PBG manifests s-long cycles of the order $O(V^{\delta})$, $\delta = 2(1-\alpha_1) < 1$, i.e. microscopic cycles (see definitions 3.5 and 3.6).

Since $\overline{\mu}_\Lambda = -C/\beta V^{\delta} + o(1/V^{\delta})$ with $\delta = 2(1-\alpha_1)$ and $C > 0$ we obtain

$$\begin{aligned}\lim_{\Gamma\uparrow\infty} \rho_{\text{long}}(\Lambda^*_{0,III,\Gamma}|\text{micro}) &:= \lim_{\Gamma\uparrow\infty} \lim_{x\downarrow 0; y\uparrow\infty} \lim_{V\uparrow\infty} \frac{1}{V} \sum_{k\in\Lambda^*_{0,III,\Gamma}} \sum_{j=xV^{\delta}}^{yV^{\delta}} \frac{1}{V} \mathrm{e}^{j\beta\overline{\mu}_\Lambda} \mathrm{e}^{-j\beta\epsilon_\Lambda(k)} \\ &= \lim_{\Gamma\uparrow\infty} \lim_{x\downarrow 0; y\uparrow\infty} \lim_{V\uparrow\infty} \sum_{k\in\Lambda^*_{0,III,\Gamma}} \frac{(\mathrm{e}^{-xC+O(1/V)} - \mathrm{e}^{-yC+O(1/V)})}{\mathrm{e}^{\beta(\epsilon_\Lambda(k)-\overline{\mu}_\Lambda)} - 1} \\ &= \lim_{\Gamma\uparrow\infty} \lim_{V\uparrow\infty} \left(\sum_{\mathbf{k}\in\Lambda^*_{0,III,\Gamma}} \rho_\Lambda(\mathbf{k})\right). \end{aligned} \quad (4.22)$$

We can easily calculate

$$\lim_{\Gamma\uparrow\infty}\lim_{V\uparrow\infty}\left(\sum_{\mathbf{k}\in\Lambda^*_{0,III,\Gamma}}\rho_\Lambda(\mathbf{k})\right)$$
$$= \lim_{\Gamma\uparrow\infty}\lim_{V\uparrow\infty}\left(\frac{1}{V^\delta}\frac{1}{V^{2\alpha_1-1}}\sum_{j=1}^{\infty}\mathrm{e}^{-(j/V^\delta)C}\sum_{n_1:|n_1/V^{2\alpha_1-1}|\leqslant\Gamma}\mathrm{e}^{-\pi\lambda_\beta^2(j/V^\delta)(n_1/V^{2\alpha_1-1})^2}\right).$$

Since this expression is nothing but the limit of double Darboux–Riemann sums, in the thermodynamical limit we obtain a double integral:

$$\lim_{\Gamma\uparrow\infty}\lim_{V\uparrow\infty}\left(\sum_{\mathbf{k}\in\Lambda^*_{0,III,\Gamma}}\rho_\Lambda(\mathbf{k})\right) = \lim_{\Gamma\uparrow\infty}\int_{\mathbb{R}^+}\mathrm{d}\zeta\,\mathrm{e}^{-\zeta C}\int_{-\Gamma}^{\Gamma}\mathrm{d}\xi\,\mathrm{e}^{-\zeta\pi\lambda_\beta^2\xi^2} = \int_{\mathbb{R}^+}\mathrm{d}\zeta\,\frac{\mathrm{e}^{-\zeta C}}{\sqrt{\zeta}\lambda_\beta} = \frac{\sqrt{\pi}}{C^{1/2}\lambda_\beta}. \tag{4.23}$$

Hence,

$$\lim_{\Gamma\uparrow\infty}\rho_{\mathrm{long}}(\Lambda^*_{0,III,\Gamma}|\mathrm{micro}) = \rho_{\mathrm{long}}(\beta,\rho) = \frac{\sqrt{\pi}}{C^{1/2}\lambda_\beta}, \tag{4.24}$$

by virtue of (4.21)–(4.23).

We know by theorem 2.2 that the density of particles in short cycles is equal to the critical density. Consequently by virtue of (4.3) and (4.24) we can conclude the proof of the theorem. □

5. Does the generalized BEC I, II, III imply a hierarchy of ODLRO?

In the introduction we presented three concepts related to BEC: g-BEC, long cycles and ODLRO. We presented in sections 2 and 3 two new concepts: *scaled* BEC (s-BEC) and *scaled* short/long cycles (s-short/long cycles) associated with g-BEC and short/long cycles. Thus it seems to be consistent to introduce a concept of *scaled* ODLRO (s-ODLRO) via our scaling approach to study the hierarchy of ODLRO.

This part gives a physical meaning of the scaling approach because it allows the study of the coherence of the condensate at large scale. We show that for very anisotropic cases ($\alpha_1 > 1/2$) the coherence length (maximal length of correlation) is not equal to the size of the box (see theorem 5.1).

5.1. Scaling approach to ODLRO

Recall that the generalized criterion of ODLRO is that there is ODLRO if and only if there is g-BEC, see theorem 2.1. The standard definition of ODLRO is formulated in definition 2.3:

$$\sigma(\beta,\rho) := \lim_{\|x-x'\|\uparrow\infty}\sigma(\beta,\rho;x,x'),$$

where $\sigma(\beta,\rho;x,x')$ is the two-point correlation function between two points x and x' after the thermodynamical limit. Note that this definition cannot be satisfactory when we use the definition of s-BEC, since we do not specify yet what are the scales of large correlations. It seems to be interesting to take the thermodynamical limit at the same time as we take the two points x and x' at increasing distance.

A natural question is whether we are able to detect different types of s-BEC (as well as g-BEC) with the help of a generalized criterion of ODLRO based on our scaling approach? We call it a *scaled* ODLRO (s-ODLRO).

Definition 5.1. *The PBG manifests a s-ODLRO if there exists a vector-valued function of volume* $X : V \mapsto X(V) \in \Lambda$ *such that* $\lim_{V\uparrow\infty} |X_\nu(V)| = \infty$, $\nu = 1, 2, 3$ *and*

$$\sigma_X(\beta, \rho) := \lim_{V\uparrow\infty} (\sigma_{\Lambda,X})(V) > 0, \tag{5.1}$$

where $(\sigma_{\Lambda,X})(V)$ *is the two-point scaled-correlation function (two-point s-correlation function) for* $x(V), x'(V) \in \Lambda$, *see (2.11):*

$$(\sigma_{\Lambda,X})(V) := \sigma_\Lambda(\beta, \rho; x(V) - x'(V)) = \sum_{k\in\Lambda^*} \rho_\Lambda(k)\, \mathrm{e}^{\mathrm{i}k\cdot X(V)}, \tag{5.2}$$

where $X(V) = (x - x')(V) \in \Lambda$.

Remark 5.1. By (2.1) and (2.2) one can write (2.11) as

$$\sigma(\beta, \rho; x, x') = \sum_{j=1}^{\infty} \mathrm{e}^{j\beta\overline{\mu}_\Lambda} \prod_{\nu=1}^{3} \theta_3\left(\frac{\pi}{V^{\alpha_\nu}}(x_\nu - x_\nu^{'}), \mathrm{e}^{-j\pi\frac{\lambda_\beta^2}{V^{2\alpha_\nu}}}\right), \tag{5.3}$$

where $\theta_3(u, q) := \sum_{n\in\mathbb{Z}^1} q^{n^2} \mathrm{e}^{2inu}$ is the elliptic theta-function.

This implies the following proposition:

Proposition 5.1. *By (5.2) the two-point correlation function as well as the two-point s-correlation function (see definition 5.1) are non-negative symmetric and* L_ν*-periodic functions of* $x_\nu - x'_\nu$, $\nu = 1, 2, 3$ *on* $\mathbb{R}$, *and decreasing/increasing on* $[nL_\nu, nL_\nu + L_\nu/2] \subset \mathbb{R}^+$, $n \in \mathbb{N}$, *respectively on* $[nL_\nu + L_\nu/2, (n + 1)L_\nu] \subset \mathbb{R}^+$, $n \in \mathbb{N}$ *(i.e., monotone on the semi-periods).*

Proof. These properties follow from the properties of the elliptic theta-function [A-S]. □

The following statement is a direct consequence of definition 2.3 and remark 5.1.

Lemma 5.1. *For any vector-valued* $X(V)$ *we have*

$$0 \leqslant \sigma_X(\beta, \rho) \leqslant \sigma(\beta, \rho). \tag{5.4}$$

This lemma means that the s-ODLRO implies standard ODLRO.

5.2. Hierarchy and anisotropy of ODLRO, coherence of the condensate

Here we use definition 5.1 to analyze s-BEC and s-long cycles in Casimir boxes. Note that the usual criterion of ODLRO is such that we have no indication of the scale of long correlations because we study their correlations after the thermodynamical limit.

We introduce a classification of s-OLDRO which is formally defined by

Definition 5.2. *The PBG manifests macroscopic-ODLRO in the direction* x_ν, *if there exists a vector* $X(V) = (X_1(V), X_2(V), X_3(V)) \in \Lambda$ *such that* $\lim_{V\uparrow\infty} |X_\nu(V)|/V^{\alpha_\nu} > 0$ *and* $\sigma_X(\beta, \rho) > 0$.

Definition 5.3. *If the PBG does not manifest macroscopic-ODLRO in the direction* x_ν *although the PBG manifests s-ODLRO, then it manifests microscopic-ODLRO in the direction* x_ν.

With periodic boundary conditions the system is homogeneous and so there is no localization of the condensate in the space contrary to the case of Dirichlet boundary conditions.

However the coherence length of the condensate could be studied on the basis of the preceding definitions 5.2 and 5.3.

Theorem 5.1. *Let us consider the grand-canonical PBG in Casimir boxes* $\Lambda = V^{\alpha_1} \times V^{\alpha_2} \times V^{\alpha_3}$ *with Dirichlet boundary conditions and a fixed density of particles* ρ. *Let* $X : V \in \mathbb{R}^+ \mapsto X(V) = (X_1(V), X_2(V), X_3(V)) \in \Lambda$, $\lim_{V\uparrow\infty} X_\nu(V) = \infty$, $0 < X_\nu(V) \leqslant V^{\alpha_\nu}/2$, $\nu = 1, 2, 3$. *Then we have the following results concerning s-ODLRO, see (5.1):*

$$\sigma_X(\beta, \rho) = 0, \qquad \text{for} \quad \rho < \rho_c(\beta). \tag{5.5}$$

Whereas for $\rho > \rho_c(\beta)$ *we obtain*

for $\alpha_1 < 1/2$*:*

$$\sigma_X(\beta, \rho) = \rho_0(\beta), \tag{5.6}$$

for $\alpha_1 = 1/2$*:*

$$\sigma_X(\beta, \rho) = \rho_0(\beta), \qquad \text{for} \quad \lim_{V\uparrow\infty} (X_1(V)/V^{\alpha_1}) = 0, \tag{5.7}$$

$$= \sum_{n_1\in\mathbb{Z}^1} \frac{\cos 2\pi n_1 x}{\pi \lambda_\beta^2 n_1^2 + B} < \rho_0(\beta), \qquad \text{for} \quad X_1(V) = xV^{\alpha_1}/2, \quad 0 < x < 1, \tag{5.8}$$

for $\alpha_1 > 1/2$*:*

$$\sigma_X(\beta, \rho) = \rho_0(\beta), \qquad \text{for} \quad \lim_{V\uparrow\infty} (X_1(V)/V^{\delta}) = 0, \quad \delta = 2(1 - \alpha_1), \tag{5.9}$$

$$= \rho_0(\beta)\, \mathrm{e}^{-2x\sqrt{\pi c}/\lambda_\beta} < \rho_0(\beta), \qquad \text{for} \quad X_1(V) = xV^{\delta/2}, \quad x > 0, \tag{5.10}$$

$$= 0, \qquad \text{for} \quad \lim_{V\uparrow\infty} (X_1(V)/V^{\delta/2}) = 0. \tag{5.11}$$

Proof. To ensure a monotonic decrease of the correlation functions for the case of periodic boundary conditions, we choose $0 < X_\nu \leqslant \frac{1}{2} V^{\alpha_\nu}$, $\nu = 1, 2, 3$ (see proposition 5.1).

The first step of the proof is to study the case $\rho < \rho_c(\beta)$.

Since $\sigma(\beta, \rho) = 0$ (theorem 2.1), by lemma 5.1 we get $\sigma_X(\beta, \rho) = 0$ for any vector $X(V) \in \Lambda$.

The second step is to study the case $\rho > \rho_c(\beta)$.

For $\alpha_1 < 1/2$ by definition 5.1 we have

$$\begin{aligned}\sigma_X(\beta, \rho) &= \lim_{V\uparrow\infty} \left(\sum_{k\in\Lambda_{0,I}^*} \rho_\Lambda(k)\, \mathrm{e}^{\mathrm{i}k.X(V)} \right) + \lim_{V\uparrow\infty} \left(\sum_{k\in\Lambda_I^*\setminus\Lambda_{0,I}^*} \rho_\Lambda(k)\, \mathrm{e}^{\mathrm{i}k.X(V)} \right) \\ &= \lim_{V\uparrow\infty} \left(\frac{1}{V} \sum_{j=1}^{\infty} \mathrm{e}^{-Aj/V} \right) + \lim_{V\uparrow\infty} \left(\sum_{k\in\Lambda_I^*\setminus\Lambda_{0,I}^*} \rho_\Lambda(k)\, \mathrm{e}^{\mathrm{i}k.X(V)} \right),\end{aligned} \tag{5.12}$$

where Λ_I^* is the dual vector space given by equation (4.1) and $\Lambda_{0,I}^* = \{k = 0\}$ is the subset corresponding to the condensate defined by (4.2). The first term of (5.12) is equal to $\rho_0(\beta, \rho)$ by virtue of theorem 4.1, so given that $\sigma(\beta, \rho) = \rho_0(\beta, \rho)$ (theorem 2.1) and by lemma 5.1 the second term of (5.12) has to be null and we get the result.

For $\alpha_1 = 1/2$ by definition 5.1 we obtain

$$\sigma_X(\beta,\rho) = \lim_{\Gamma\uparrow\infty}\lim_{V\uparrow\infty}\left(\sum_{k\in\Lambda^*_{0,II,\Gamma}}\rho_\Lambda(k)\,\mathrm{e}^{\mathrm{i}k.X(V)}\right) + \lim_{\Gamma\uparrow\infty}\lim_{V\uparrow\infty}\left(\sum_{k\in\Lambda^*_{II}\backslash\Lambda^*_{0,II,\Gamma}}\rho_\Lambda(k)\,\mathrm{e}^{\mathrm{i}k.X(V)}\right)$$
$$= \lim_{\Gamma\uparrow\infty}\lim_{V\uparrow\infty}\left(\frac{1}{V}\sum_{j=1}^{\infty}\mathrm{e}^{-Bj/V}\sum_{n_1=-\Gamma}^{\Gamma}\mathrm{e}^{-\pi\lambda_\beta^2 n_1^2(j/V)}\,\mathrm{e}^{2\pi\mathrm{i}X_1(V)(n_1/V)}\right)$$
$$+ \lim_{\Gamma\uparrow\infty}\lim_{V\uparrow\infty}\left(\sum_{k\in\Lambda^*_{II}\backslash\Lambda^*_{0,II,\Gamma}}\rho_\Lambda(k)\,\mathrm{e}^{\mathrm{i}k.X(V)}\right), \tag{5.13}$$

where Λ^*_{II} is the dual vector space given by equation (4.10) and $\Lambda^*_{0,II,\Gamma}$ is the subset corresponding to the condensate defined by (4.11).

If $\lim_{V\uparrow\infty}(X_1(V)/V^{\alpha_1}) = 0$, the first term in the right-hand side of (5.13) is equal to $\rho_0(\beta,\rho)$ by virtue of theorem 4.2. Given that $\sigma(\beta,\rho) = \rho_0(\beta,\rho)$ (theorem 2.1) and by lemma 5.1, the second term in (5.13) is null and we obtain the result.

Let $\lim_{V\uparrow\infty}(X_1(V)/V^{\alpha_1}) = x$, $0 < x \leqslant 1/2$. Since the sum inside the limit in the first term of the right-hand side of (5.13) is a Darboux–Riemann sum, one obtains

$$\int_{\mathbb{R}^+}\mathrm{d}\chi\,\mathrm{e}^{-B\chi}\sum_{n_1\in\mathbb{Z}^1}\mathrm{e}^{-\pi\lambda_\beta^2 n_1^2\chi}\,\mathrm{e}^{2\pi ixn_1} = \sum_{n_1\in\mathbb{Z}^1}\frac{\cos 2\pi n_1 x}{\pi\lambda_\beta^2 n_1^2 + B}.$$

The second term in the right-hand side of (5.13) is null because the phase implies that it is smaller than the density of particles in $\Lambda^*_{II}\backslash\Lambda^*_{II,0,\Gamma}$ which is null in the limit $\Gamma\to\infty$.

For $\alpha_1 > 1/2$ by definition 5.1 and by virtue of (4.23) we have

$$\sigma_X(\beta,\rho) = \lim_{\Gamma\uparrow\infty}\lim_{V\uparrow\infty}\left(\sum_{k\in\Lambda^*_{0,III,\Gamma}}\rho_\Lambda(k)\,\mathrm{e}^{\mathrm{i}k.X(V)}\right) + \lim_{\Gamma\uparrow\infty}\lim_{V\uparrow\infty}\left(\sum_{k\in\Lambda^*_{III}\backslash\Lambda^*_{0,III,\Gamma}}\rho_\Lambda(k)\,\mathrm{e}^{\mathrm{i}k.X(V)}\right)$$
$$= \lim_{\Gamma\uparrow\infty}\lim_{V\uparrow\infty}\left(\frac{1}{V^\delta}\frac{1}{V^{2\alpha_1-1}}\sum_{j=1}^{\infty}\mathrm{e}^{-Cj/V^\delta}\sum_{n_1:|n_1/V^{2\alpha_1-1}|\leqslant\Gamma}\mathrm{e}^{-\pi\lambda_\beta^2(j/V^\delta)(n_1/V^{2\alpha_1-1})^2}\,\mathrm{e}^{2\pi iX_1(V)n_1/V^{\alpha_1}}\right)$$
$$+ \lim_{\Gamma\uparrow\infty}\lim_{V\uparrow\infty}\left(\sum_{k\in\Lambda^*_{III}\backslash\Lambda^*_{0,III,\Gamma}}\rho_\Lambda(k)\,\mathrm{e}^{\mathrm{i}k.X(V)}\right), \tag{5.14}$$

where Λ^*_{III} is the dual vector space given by equation (4.18) and $\Lambda^*_{0,III}$ is the subset corresponding to the condensate defined by (4.19).

If $\lim_{V\uparrow\infty}(X_1(V)/V^{\delta/2}) = 0$, the right-hand side of (5.14) is equal to $\rho_0(\beta,\rho)$ by virtue of theorem 4.3. Since $\sigma(\beta,\rho) = \rho_0(\beta,\rho)$ (theorem 2.1), by lemma 5.1 the second term of (5.14) has to be null and we get the result.

Let $\lim_{V\uparrow\infty}(X_1(V)/V^{\delta/2}) = x$, $x > 0$, $\delta = 2(1-\alpha_1)$. Then the sum inside the limit in the first term of (5.14) is a double Darboux–Riemann sum, which implies

$$\int_{\mathbb{R}^+}\mathrm{d}\xi\,\mathrm{e}^{-C\xi}\int_{\chi\in\mathbb{R}}\mathrm{d}\chi\,\mathrm{e}^{-\pi\lambda_\beta^2\xi\chi^2}\,\mathrm{e}^{2\pi ix\chi} = \frac{\sqrt{\pi}}{\lambda_\beta\sqrt{C}}\,\mathrm{e}^{-2x\sqrt{\pi c}/\lambda_\beta}.$$

By the same argument as in the case $\alpha_1 = 1/2$ the second part of (5.14) is null (the preceding expression is a decreasing function of x for $x > 0$) and thus we obtain the result.

Let $\lim_{V\uparrow\infty}(X_1(V)/V^{\delta/2}) = \infty$. Since the correlation function is a decreasing function for $0 < X_\nu \leqslant V^{\alpha_\nu}/2$ (see proposition 5.1), it is uniformly bounded by the above estimate with $X(V) = xV^\delta$, $x > 0$:

$$\int_{\mathbb{R}^+} \mathrm{d}\xi\, \mathrm{e}^{-C\xi} \int_{\chi\in\mathbb{R}} \mathrm{d}\chi\, \mathrm{e}^{-\pi\lambda_\beta^2\xi\chi^2}\, \mathrm{e}^{2\pi\mathrm{i}x\chi} + \lim_{\Gamma\uparrow\infty}\lim_{V\uparrow\infty}\left(\sum_{k\in\Lambda_{III}^*\setminus\Lambda_{0,III,\Gamma}^*} \rho_\Lambda(k)\,\mathrm{e}^{\mathrm{i}k.X(V)}\right).$$

When x tends to infinity, the first part goes to zero (by the Riemann–Lesbegue theorem) then the preceding arguments show that the second part is also null. This concludes the proof. □

We give here a classification of s-ODLRO for three cases of Casimir boxes:

Theorem 5.2. *If one takes the Casimir boxes with $\alpha \leqslant 1/2$ then for a fixed density $\rho > \rho_c(\beta)$ the PBG manifests macroscopic-ODLRO in three directions. If $\alpha > 1/2$ then for a fixed density $\rho > \rho_c(\beta)$ the PBG manifests microscopic-ODLRO in direction x_ν and macroscopic-ODLRO in other directions.*

Proof. Definitions 5.2, 5.3 and theorem 5.1 give the proof of the theorem. □

It is remarkable that for types I and II g-BEC in Casimir boxes corresponding to $\alpha_1 \leqslant 1/2$ the condensate is spatially macroscopic whereas for the case $\alpha_1 > 1/2$ the condensate is spatially macroscopic in two directions but microscopic in the most anisotropic direction x_1. It is natural to guess that there is a link between the size of s-long cycles and coherence length of the condensate. We can see this explicitly in [U] where the competition between the size of correlation X and the size of cycle j indicates that the coherence length is of the order of the square root of the size of the s-long cycles (e.g., $V^{\delta/2}$, $\delta = 2(1-\alpha_1)$ in the case of the PBG in Casimir boxes with $\alpha_1 > 1/2$).

6. Concluding remarks

6.1. Some technical remarks

In this paper we introduce a new concept of BEC which is called scaled BEC (s-BEC) to adapt the London scaling approach to the problem of g-BEC. It implies the van den Berg–Lewis–Pulé classification of BEC in three types (I, II, III) illustrated for the particular case of the PBG in Casimir boxes. This is a first formal step necessary before studying more carefully the different case of g-BEC for the PBG in Casimir boxes. One can see this by virtue of proposition 3.1.

The fundamental question that we study in this paper is the relation between different types of g-BEC (I, II, III) with long cycles and with ODLRO. Our results concerning the PBG in Casimir boxes can be summarized as follows:

- We introduced a new concept of short/long cycles called scaled short/long cycles (s-short/long cycles, see definition 3.2) to distinguish different types of g-BEC, see theorems 4.1–4.3, remark 3.2 and proposition 3.3. This paper is based on the estimation of the size of s-long cycles in the condensate, see definitions 3.3 and 3.5. If the size of s-long cycles is macroscopic, then the g-BEC is of type I or II but if the s-long cycles are microscopic, then the g-BEC is of type III.
- We introduced a new concept of ODLRO, called scaled ODLRO (s-ODLRO, see definition 5.1), to distinguish the different types of g-BEC, see theorems 5.1 and 5.2. Our arguments are based on the estimate of the coherence length of the condensate, see

definitions 5.2 and 5.3. If the coherence length is macroscopic in the three directions, then the g-BEC is of type I or II, and if in one of the three dimensions the coherence length is microscopic, then the g-BEC is of type III.

It is clear that the proof of theorems 4.1–4.3 is based on an analysis of geometric series easily done via s-long cycles. For this reason we can say that s-long cycles is a well adapted technique to study the classification of BEC. Another reason is that the concept of cycles is independent of the representation of the gas (Feynman–Kac versus spectral representation). Therefore the cycles seem to be useful to study generalized BEC for an interacting Bose gas.

For simplicity we consider here the PBG with periodic boundary conditions but one can adapt the present paper to Dirichlet or Neumann boundary conditions for which we can characterize the geometric form of the condensate cloud via the concept of scaled local density. In these cases the results concerning the hierarchies of cycles and ODLRO do not change. However, if one takes attractive boundary conditions, see [V-V-Z], we guess that the result should be different, since in this case the condensate is localized in two modes and it is not homogeneous. One can suppose that the s-long cycles are macroscopic but the most interesting feature is the coherence length of the condensate and its geometric form.

The last technical remark concerns the validity of the results in the canonical ensemble. It is known [P-Z] that in the canonical ensemble for the PBG in Casimir boxes there is the same type of generalized BEC as in the grand-canonical ensemble for the equivalent geometry. Thus we believe that the principal conclusions concerning the hierarchy of long cycles and of ODLRO should not change. However given that the amount of condensate in individual states is different [P-Z] from the grand-canonical case, some results (formulae in the propositions 3.1 and 3.3 and theorem 5.1) should be different.

6.2. Conceptual and physical remarks

First note that ODLRO, long cycles and generalized BEC are equivalent criteria for PBG in Casimir boxes. Consequently generalized BEC is more relevant than usual BEC (macroscopical occupation of the ground-state) because it contains all cases of BEC (fragmented or not).

In this paper we present a scaling approach for Bose–Einstein condensation. It allows the classification of different types of condensate via the scaling size of long cycles in relation with the scaling size of large correlations. The interest in the study of long cycles using the scaling approach is the knowledge on coherence properties of the condensate by virtue of the rule of Bose statistics in the two-point correlation function. Heuristically the order of the size of large correlation is the square root of the order of the size of long cycles.

The scaling approach should be useful in the interpretation of thermodynamical results for large particle number systems. It is interesting because one can study the effect of the geometry of the box on the geometry and coherence properties of the condensate. In this paper, we show that a condensate is like a finite or infinite number of macroscopic particles (type I or II) or infinite number of microscopic particles (type III, analogous to quasi-condensate [M-H-U-B, P-S-W]) formed by 'closed polymer chain' (cycles) of macroscopic or microscopic size related to the coherence length (square root of size of cycles). Physically it could be interesting to study the correlation function of a condensate in a harmonic trap with pulsations $\omega_x, \omega_y, \omega_z$ using our scaling approach (e.g., for very anisotropic traps).

6.3. Perspectives

In the present paper the choice of Casimir boxes serves to illustrate our concepts. But we can use *van den Berg* boxes, which are a generalization of Casimir ones ($\Lambda_L =$

$L_1(L) \times L_2(L) \times L_3(L)$ with $V_L := |\Lambda_L| = L_1(L)L_2(L)L_3(L)$ where $L_i(L)$ are functions of a parameter L such as $\lim_{L\to\infty} L_i(L) = \infty$). These boxes are very interesting to study since with a particular choice of the functions $L_i(L)$, e.g. (see [vdB]) $L_1(L) = L_2(L) = \mathrm{e}^L$, $L_3(L) = L$ with Dirichlet boundary conditions, for $\rho_c(\beta) < \rho < \rho_m(\beta)$ the g-BEC is of type III and for $\rho > \rho_m(\beta)$ it seems there is the *coexistence* of g-BEC of type I and type III, where $\rho_c(\beta)$ is the critical density defined by (2.5) and $\rho_m(\beta)$ is a critical density defined in this particular case by $\rho_m(\beta) = \rho_c(\beta) + 1/\beta\pi$. In fact $\rho_c(\beta)$ separates condensate and non-condensate regimes and $\rho_m(\beta)$ separates the different types of generalized BEC. This seems to be analogous to a quasi-condensate/condensate transition [P-S-W]. It is natural now to study this curious phenomenon using our approach, which will be a subject of another paper.

We now discuss whether the scaling approach for the interacting gas problem is still relevant. One can see that the concept of g-BEC is well formulated for the PBG but not for the interacting Bose gas. Then how do we study the classification of BEC for the interacting Bose gas if we have no more indications about the spectral properties of the gas? This problem could be solved using our scaling approach (of long cycles or ODLRO). Thus the next step will be the application of these new methods for some models of interacting Bose gas (especially for weakly interacting Bose gas at first time, take e.g. an interaction $U = \sum_{k\in\Lambda^*} gN_k^2/2V$, $g > 0$, see [Br-Z])). For strongly interacting bosons in *quantum crystals*, there is a formation of infinite cycles [U2]. It could be interesting to study them using the scaling approach of long cycles.

Another interesting problem is the characterization of quantum fluctuations in many-body systems. The scaling approach could also be relevant to study the finite size scaling effect [B-D-T] appearing in some systems, such as the Casimir effect in quantum liquids [M-Z, Z-al].

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